

Reconciling Lambek’s Restriction, Cut-Elimination, and Substitution in the Presence of Exponential Modalities

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Abstract

The Lambek calculus can be considered as a version of non-commutative intuitionistic linear logic. One of the interesting features of the Lambek calculus is the so-called “Lambek’s restriction,” that is, the antecedent of any provable sequent should be non-empty. In this paper we discuss ways of extending the Lambek calculus with the linear logic exponential modality while keeping Lambek’s restriction. Interestingly enough, we show that for any system equipped with a reasonable exponential modality the following holds: if the system enjoys cut elimination and substitution to the full extent, then the system necessarily violates Lambek’s restriction. Nevertheless, we show that two of the three conditions can be implemented. Namely, we design a system with Lambek’s restriction and cut elimination and another system with Lambek’s restriction and substitution. For both calculi we prove that they are undecidable, even if we take only one of the two divisions provided by the Lambek calculus. The system with cut elimination and substitution and without Lambek’s restriction is folklore and known to be undecidable.

1 Introduction

1.1 The Lambek Calculus

The Lambek calculus was introduced by J. Lambek in [13] for mathematical description of natural language syntax by means of so-called *Lambek categorial (type-logical) grammars* (see, for example, [4][19][17]). In Lambek grammars, syntactic categories are represented by logical formulae involving three connectives: the *product* (corresponds to concatenation of words) and two *divisions* (left and right), and syntactic correctness of natural language expressions corresponds to derivability in the Lambek calculus.

For simplicity, in this paper we discuss only the product-free fragment of the Lambek calculus. First we consider not the Lambek calculus \mathbf{L} [13], but its variant \mathbf{L}^* [14]. The difference between \mathbf{L} and \mathbf{L}^* is explained in the end of this introductory section (see “Lambek’s Restriction”).

\mathbf{L}^* is a substructural logic, and here we formulate it as a Gentzen-style sequent calculus. Formulae of \mathbf{L}^* are called *types* and are built from *variables*, or *primitive types* (p, q, r, p_1, p_2, \dots) using two binary connectives: \backslash (*left division*) and $/$ (*right division*). Types are denoted by capital Latin letters; finite (possibly empty) linearly ordered sequences of types by capital Greek ones. Λ stands for the empty sequence. The Lambek calculus derives objects called *sequents* of the form $\Pi \rightarrow A$, where the *antecedent* Π is a linearly ordered sequence of types and the *succedent* A is a type.

The axioms of \mathbf{L}^* are all sequents $A \rightarrow A$, where A is a type, and the rules of inference are as follows:

$$\begin{array}{c} \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \backslash B} (\rightarrow \backslash) \quad \frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, \Pi, A \backslash B, \Delta_2 \rightarrow C} (\backslash \rightarrow) \\[1em] \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} (\rightarrow /) \quad \frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, B / A, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow) \end{array}$$

For \mathbf{L}^* and other calculi introduced later in this paper, we do not include cut as an official rule of the system. However, the cut rule of the following non-commutative form

$$\frac{\Pi \rightarrow A \quad \Delta_1, A, \Delta_2 \rightarrow B}{\Delta_1, \Pi, \Delta_2 \rightarrow B} (\text{cut})$$

is admissible in \mathbf{L}^* [14].

By \mathbf{L}^*_\backslash (resp., $\mathbf{L}^*_/\backslash$) we denote the fragment of \mathbf{L}^* with only the right (resp., left) division connective. Due to the subformula property, these fragments are obtained from the full calculus simply by restricting the set of rules.

1.2 The Exponential Modality

We see that \mathbf{L}^* lacks structural rules (except for the implicit rule of associativity).

\mathbf{L}^* can be conservatively embedded [1][27] into a non-commutative, intuitionistic or cyclic, variant of Girard's [5] linear logic. In the spirit of linear logic connectives, the Lambek calculus can be extended with the *exponential* unary connective that enables structural rules (weakening, contraction, and commutativity) in a controlled way.

We'll denote this extended calculus by \mathbf{EL}^* . Types of \mathbf{EL}^* are built from variables using two binary connectives (\backslash and $/$) and a unary one, $!$, called the *exponential*, or, colloqually, "*bang*." If $\Gamma = A_1, \dots, A_k$, then by $!\Gamma$ we denote the sequence $!A_1, \dots, !A_k$. \mathbf{EL}^* is obtained from \mathbf{L}^* by adding the following rules:

$$\begin{array}{ll}
\frac{\Delta_1, A, \Delta_2 \rightarrow B}{\Delta_1, !A, \Delta_2 \rightarrow B} (! \rightarrow) & \frac{!\Gamma \rightarrow A}{!\Gamma \rightarrow !A} (\rightarrow !) \\
\frac{\Delta \rightarrow B}{!A, \Delta \rightarrow B} (\text{weak}) & \frac{!A, !A, \Delta \rightarrow B}{!A, \Delta \rightarrow B} (\text{contr}) \\
\frac{\Delta_1, B, !A, \Delta_2 \rightarrow C}{\Delta_1, !A, B, \Delta_2 \rightarrow C} (\text{perm}_1) & \frac{\Delta_1, !B, A, \Delta_2 \rightarrow C}{\Delta_1, A, !B, \Delta_2 \rightarrow C} (\text{perm}_2)
\end{array}$$

The following theorem is proved in [9] and [6] and summarized in [8]. A weaker result that \mathbf{EL}^* with the product and two divisions is undecidable follows from [15].

Theorem 1. *The derivability problem for \mathbf{EL}^* is undecidable.*

1.3 Lambek's Restriction

The original Lambek calculus \mathbf{L} [13] differs from the presented above in one detail: in \mathbf{L} , sequents with empty antecedents are not permitted. This restriction applies not only to the final sequent, but to all ones in the derivation. Thus, for example, the sequent $(q \backslash q) \backslash p \rightarrow p$ is derivable in \mathbf{L}^* , but not in \mathbf{L} , though its antecedent is not empty (but the \mathbf{L}^* -derivation involves the sequent $\rightarrow q \backslash q$ with an empty antecedent). Further we shall use the term *Lambek's restriction* for this special constraint. Actually, Lambek's restriction in \mathbf{L}^* could potentially be violated only by application of the $(\rightarrow \backslash)$ and $(\rightarrow /)$ rules, therefore \mathbf{L} can be obtained from \mathbf{L}^* by adding the constraint " Π is non-empty" to these two rules.

At first glance, Lambek's restriction looks strange and formal, but it is highly motivated by linguistic applications.

Example 1. [17, 2.5] In syntactic formalisms based on the Lambek calculus, Lambek types denote syntactic categories. Let n stand for “noun phrase,” then n/n is going to be a “noun modifier” (it can be combined with a noun phrase on the right producing a new, more complex noun phrase: $\mathbf{L} \vdash n/n, n \rightarrow n$), i.e. an adjective. Adverbs, as adjective modifiers, receive the type $(n/n)/(n/n)$. Now one can derive the sequent $(n/n)/(n/n), n/n, n \rightarrow n$ and therefore establish that, say, “very interesting book” is a valid noun phrase (belongs to syntactic category n). However, in \mathbf{L}^* one can also derive $(n/n)/(n/n), n \rightarrow n$, where the antecedent describes syntactic constructions like “very book,” that in fact aren’t correct noun phrases.

This example shows that, for linguistic purposes, \mathbf{L} is more appropriate than \mathbf{L}^* .

Suprisingly, however, it is not so straightforward to add the exponential to \mathbf{L} or to impose Lambek’s restriction on \mathbf{EL}^* . In Sections 2–7 we discuss several ways how to do this, define a number of the corresponding calculi, prove their properties, and discuss some issues connected with these calculi.

In Section 9 we state and prove undecidability results for calculi defined earlier; in Section 8 we prepare the techniques then used in Section 9. Finally, Section 10 contains general discussion of the results and possible directions of future work.

1.4 Is it Possible to Maintain Three Properties Together: Lambek’s Restriction, Cut Elimination, and Substitution?

No. We show (Theorems 3 and 4) that for any system equipped a reasonable ! the following holds: if the system enjoys cut elimination and substitution in full extent, then this system necessarily violates Lambek’s restriction. (More precisely, adding one formula starting with ! to the antecedent allows \mathbf{L}^* derivations inside such a system.)

Nevertheless, any two of these three properties are realisable in the calculi defined below, namely:

- \mathbf{EL}^- (Section 3) has Lambek’s restriction and cut elimination, but substitution only for formulae without !;
- \mathbf{EL}^{mk} (Section 6) has Lambek’s restriction and substitution in the full form, but the cut rule is admissible only for formulae without !;
- finally, \mathbf{EL}^* enjoys both substitution and cut, but without Lambek’s restriction.

2 Imposing Lambek's Restriction on \mathbf{EL}^* : the 1st Approach, \mathbf{EL}^{wk}

The first, naïve way of imposing Lambek's restriction on \mathbf{EL}^* is to restrict only rules $(\rightarrow \backslash)$ and $(\rightarrow /)$ in the same way as it is done in \mathbf{L} . Notice that all other rules, including rules for the exponential, preserve the non-emptiness of the antecedent. Denote the calculus by \mathbf{EL}^{wk} .

However, such a restriction doesn't change things significantly, since the following lemma provides the non-emptiness of the antecedent for free:

Lemma 2.1. *Let p be a variable not occurring in a sequent $\Gamma \rightarrow A$. Then*

$$\mathbf{EL}^* \vdash \Gamma \rightarrow A \iff \mathbf{EL}^{\text{wk}} \vdash !p, \Gamma \rightarrow A.$$

This lemma shows that \mathbf{EL}^* -derivations can be enabled in \mathbf{EL}^{wk} by an easy technical trick. Therefore, Theorem 1 implies immediately that \mathbf{EL}^{wk} is undecidable.

Lemma 2.2. $\mathbf{EL}^{\text{wk}} \vdash !B, \Gamma \rightarrow A \iff \mathbf{EL}^* \vdash !B, \Gamma \rightarrow A.$

These two lemmas are proved by induction on the derivations (recall that (cut) is not included in the calculi).

Thus, Lambek's restriction in \mathbf{EL}^{wk} vanishes as soon as the antecedent contains a formula with $!$ as the main connective. And, unfortunately, this acts non-locally: once $!A$ appears somewhere in the antecedent, one can freely derive unwanted things like “very book” (see Example 1 above).

3 Imposing Lambek's Restriction on \mathbf{EL}^* : the 2nd Approach, \mathbf{EL}^-

To overcome the ability of $!B$ to mimic the empty antecedent, we impose more radical restrictions by constructing the following calculus \mathbf{EL}^- .

Any formula not of the form $!B$ is called a *non-bang-formula*. (A non-bang-formula is allowed to have proper subformulae with $!$.) Now \mathbf{EL}^- is defined by the following axioms and rules:

$$A \rightarrow A$$

$$\frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \backslash B} (\rightarrow \backslash), \text{ where } \Pi \text{ contains a non-bang-formula}$$

$$\frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} (\rightarrow /), \text{ where } \Pi \text{ contains a non-bang-formula}$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \rightarrow C} (\setminus \rightarrow) \quad \frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, B / A, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow)$$

$$\frac{\Delta_1, A, \Delta_2 \rightarrow B}{\Delta_1, !A, \Delta_2 \rightarrow B} (! \rightarrow), \text{ where } \Delta_1, \Delta_2 \text{ contains a non-bang-formula}$$

$$\frac{\Delta \rightarrow B}{!A, \Delta \rightarrow B} (\text{weak}) \quad \frac{!A, !A, \Delta \rightarrow B}{!A, \Delta \rightarrow B} (\text{contr})$$

$$\frac{\Delta_1, B, !A, \Delta_2 \rightarrow C}{\Delta_1, !A, B, \Delta_2 \rightarrow C} (\text{perm}_1) \quad \frac{\Delta_1, !B, A, \Delta_2 \rightarrow C}{\Delta_1, A, !B, \Delta_2 \rightarrow C} (\text{perm}_2)$$

Note that in the $(\rightarrow !)$ rule of \mathbf{EL}^* all the formulae in the antecedent are of the form $!B$. Therefore there is no $(\rightarrow !)$ rule in \mathbf{EL}^- . Also note that the cut rule is not officially included in \mathbf{EL}^- ; in the next section we prove that it is admissible.

Lemma 3.1. *If $\Pi \rightarrow A$ is derivable in \mathbf{EL}^- and Π is of the form $!\Gamma$, then A is of the form $!B$, such that $!B$ appears in $!\Gamma$.*

Proof. A sequent with no non-bang-formula in the antecedent can be obtained only by applying (weak), (perm), (contr) (in any combination) to an axiom of the form $!B \rightarrow !B$. All other rules either introduce non-bang-formulae to the left, or explicitly require their existence. \square

Now Lambek's restriction in \mathbf{EL}^- is stated in the following way: *in a non-trivial derivable sequent $\Pi \rightarrow A$ the antecedent Π should contain at least one non-bang-formula.*

4 The Cut Rule in \mathbf{EL}^-

Lemma 4.1. *If $\Pi \rightarrow A$ is derivable in \mathbf{EL}^- and A is a non-bang-formula, then Π necessarily contains a non-bang-formula.*

Proof. Immediately from Lemma 3.1. □

Theorem 2. *The cut rule*

$$\frac{\Pi \rightarrow A \quad \Delta_1, A, \Delta_2 \rightarrow C}{\Delta_1, \Pi, \Delta_2 \rightarrow C} \text{ (cut)}$$

is admissible in \mathbf{EL}^- .

Proof. We proceed by double induction. We consider a number of cases, and in each of them the cut either disappears, or is replaced by cuts with simpler cut formulae (A), or is replaced by a cut for which the depth of at least one derivation tree of a premise ($\Pi \rightarrow A$ or $\Delta_1, B, \Delta_2 \rightarrow C$) is less than for the original cut, and the other premise derivation and the cut formula remain the same. Thus by double induction (on the outer level—on the complexity of A , on the inner level—on the sum of premise derivation tree depths) we get rid of the cut.

Case 1: One of the premises of the cut rule is the axiom (a sequent of the form $A \rightarrow A$). Then the cut disappears, since its other premise coincides with the goal sequent.

Case 2: A is not the formula that is introduced by the lowermost rule in the derivation of one of the premises of the cut. (The term “*formula introduced by a rule*” here means the following: rules $(/ \rightarrow)$, $(\backslash \rightarrow)$, $(\rightarrow /)$, $(\rightarrow \backslash)$, $(! \rightarrow)$, and $(\rightarrow !)$ introduce the formula that includes the new connective; (weak) and (contr) introduce $!A$ involved in these rules; (perm₁) and (perm₂) do not introduce anything.)

In this case (cut) can be interchanged with that lowermost rule. Many subcases arise here, depending on the particular form of the rule interchanged with (cut), but they are all handled similarly. Below we show only the most interesting situations, when we interchange $(! \rightarrow)$ or $(\rightarrow /)$ ($(\rightarrow \backslash)$ is symmetric) with (cut). In these transformations Lambek’s restriction imposed on these rules could potentially get violated after the exchange with (cut) (we show that this doesn’t happen).

Transformation 1.

$$\frac{\frac{\Pi', D, \Pi'' \rightarrow A}{\Pi', !D, \Pi'' \rightarrow A} (! \rightarrow) \quad \Delta_1, A, \Delta_2 \rightarrow C}{\Delta_1, \Pi', !D, \Pi'', \Delta_2 \rightarrow C} \text{ (cut)}$$

$$\Downarrow$$

$$\frac{\Pi', D, \Pi'' \rightarrow A \quad \Delta_1, A, \Delta_2 \rightarrow C}{\Delta_1, \Pi', D, \Pi'', \Delta_2 \rightarrow C} \text{ (cut)}$$

$$\frac{\Delta_1, \Pi', D, \Pi'', \Delta_2 \rightarrow C}{\Delta_1, \Pi', !D, \Pi'', \Delta_2 \rightarrow C} (! \rightarrow)$$

Transformation 2.

$$\begin{array}{c}
\frac{\Pi \rightarrow A \quad \frac{\Delta_1, A, \Delta'_2, D, \Delta''_2 \rightarrow C}{\Delta_1, A, \Delta'_2, !D, \Delta''_2 \rightarrow C} (! \rightarrow)}{\Delta_1, \Pi, \Delta'_2, !D, \Delta''_2 \rightarrow C} (\text{cut}) \\
\Downarrow \\
\frac{\Pi \rightarrow A \quad \Delta_1, A, \Delta'_2, D, \Delta''_2 \rightarrow C}{\Delta_1, \Pi, \Delta'_2, D, \Delta''_2 \rightarrow C} (\text{cut}) \\
\frac{\Delta_1, \Pi, \Delta'_2, D, \Delta''_2 \rightarrow C}{\Delta_1, \Pi, \Delta'_2, !D, \Delta''_2 \rightarrow C} (! \rightarrow)
\end{array}$$

Transformation 3.

$$\begin{array}{c}
\frac{\Pi \rightarrow A \quad \frac{\Delta_1, A, \Delta_2, C_1 \rightarrow C_2}{\Delta_1, A, \Delta_2 \rightarrow C_2 / C_1} (\rightarrow /)}{\Delta_1, \Pi, \Delta_2 \rightarrow C_2 / C_1} (\text{cut}) \\
\Downarrow \\
\frac{\Pi \rightarrow A \quad \Delta_1, A, \Delta_2, C_1 \rightarrow C_2}{\Delta_1, \Pi, \Delta_2, C_1 \rightarrow C_2} (\text{cut}) \\
\frac{\Delta_1, \Pi, \Delta_2, C_1 \rightarrow C_2}{\Delta_1, \Pi, \Delta_2 \rightarrow C_2 / C_1} (\rightarrow /)
\end{array}$$

In Transformation 1, the existence of a non-bang-formula in Π' or Π'' implies its existence in the larger context $\Delta_1, \Pi', \Pi'', \Delta_2$, thus application of $(! \rightarrow)$ is legal. For Transformations 2 and 3, if the non-bang-formula guaranteed by Lambek's restriction (which was indeed valid before the transformation) is A itself, then a non-bang-formula also appears in Π by Lemma 4.1. Otherwise we can take the same non-bang-formula as before the transformation.

Case 3: A is introduced by the lowermost rules both into $\Pi \rightarrow A$ and into $\Delta_1, A, \Delta_2 \rightarrow C$. Note that in \mathbf{EL}^- there is no rule that introduces a formula of the form $!E$ to the succedent. Therefore A is either of the form E / F , or $F \setminus E$. We consider only the former, the latter is handled symmetrically. The derivation is transformed in the following way:

Transformation 4.

$$\frac{\frac{\Pi, F \rightarrow E}{\Pi \rightarrow E / F} (\rightarrow /) \quad \frac{\Gamma \rightarrow F \quad \Delta_1, E, \Delta_2 \rightarrow C}{\Delta_1, E / F, \Gamma, \Delta_2 \rightarrow C} (/ \rightarrow)}{\Delta_1, \Pi, \Gamma, \Delta_2 \rightarrow C} (\text{cut})$$

$$\begin{array}{c} \Downarrow \\ \frac{\frac{\Gamma \rightarrow F \quad \Pi, F \rightarrow E}{\Pi, \Gamma \rightarrow E} \text{ (cut)} \quad \Delta_1, E, \Delta_2 \rightarrow C}{\Delta_1, \Pi, \Gamma, \Delta_2 \rightarrow C} \text{ (cut)} \end{array}$$

The new cut formulae, E and F , are simpler than the original one, E / F . \square

5 Substitution Issues

\mathbf{EL}^- inherits a bang-free substitution lemma: derivability of a sequent is preserved if we replace all occurrences of a variable q with a formula Q without $!$.

In the general case, however, \mathbf{EL}^- doesn't respect type substitution. For instance, $p, !(p \setminus q) \rightarrow q$ is derivable in \mathbf{EL}^- , but $!r, !(r \setminus q) \rightarrow q$ is not. Unfortunately, this is not just a problem with this particular system, but a general issue: as we show below, in the presence of the exponential modality any system enjoying admissibility of (cut) and general substitution lemma necessarily violates Lambek's restriction.

Let \mathbf{EL}^\dagger be an arbitrary calculus, in the same language as \mathbf{EL}^* , satisfying the properties below. (Note that these properties do not *define* the calculus in a unique way—we rather talk about *a family* of possible 'good' extensions of the Lambek calculus.)

1. *Extension.* If a sequent is derivable in \mathbf{L} , then it is derivable in \mathbf{EL}^\dagger .
2. *Cut.* The cut rule of the form

$$\frac{\Pi \rightarrow A \quad \Delta_1, A, \Delta_2 \rightarrow C}{\Delta_1, \Pi, \Delta_2 \rightarrow C} \text{ (cut)}$$

is admissible in \mathbf{EL}^\dagger .

3. *Substitution.* The following rule is admissible in \mathbf{EL}^\dagger :

$$\frac{\Pi \rightarrow A}{\Pi[q := Q] \rightarrow A[q := Q]} \text{ (subst)}$$

Here q is a variable, Q is a formula (possibly with $!$), and $[q := Q]$ denotes substitution of Q for q .

4. *Monotonicity.* The following rules are admissible in \mathbf{EL}^\dagger :

$$\frac{A_1 \rightarrow A_2 \quad B_1 \rightarrow B_2}{B_1 / A_2 \rightarrow B_2 / A_1} (\text{mon}/)$$

$$\frac{A_1 \rightarrow A_2 \quad B_1 \rightarrow B_2}{A_2 \setminus B_1 \rightarrow A_1 \setminus B_2} (\text{mon}\setminus)$$

5. *Weakening, contraction, and permutation.* The following rules are admissible in \mathbf{EL}^\dagger :

$$\frac{\Delta \rightarrow B}{!A, \Delta \rightarrow B} (\text{weak}) \quad \frac{!A, !A, \Delta \rightarrow B}{!A, \Delta \rightarrow B} (\text{contr})$$

$$\frac{\Delta_1, B, !A, \Delta_2 \rightarrow C}{\Delta_1, !A, B, \Delta_2 \rightarrow C} (\text{perm}_1) \quad \frac{\Delta_1, !B, A, \Delta_2 \rightarrow C}{\Delta_1, A, !B, \Delta_2 \rightarrow C} (\text{perm}_2)$$

6. The rules

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, B / A, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow) \quad \text{and} \quad \frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \rightarrow C} (\setminus \rightarrow)$$

are admissible in \mathbf{EL}^\dagger without restrictions.

7. If Π contains a formula without occurrences of $!$ (and therefore is non-empty) and B does not contain occurrences of $!$, then the rules

$$\frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} (\rightarrow /) \quad \text{and} \quad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} (\rightarrow \setminus)$$

are admissible in \mathbf{EL}^\dagger .

Note that (cut), (subst), and (mon) are admissible in \mathbf{L} , therefore we want them to keep valid in the extension. Weakening, contraction, and permutation are basic rules for the exponential. Finally, the last two properties ensure that the version of Lambek's restriction used in \mathbf{EL}^\dagger does not forbid Lambek derivations in the presence of the exponential modality.

Also note that by substitution we get the axiom $A \rightarrow A$ for arbitrary A , possibly with occurrences of $!$.

Unfortunately, any calculus \mathbf{EL}^\dagger with these 7 properties necessarily violates Lambek's restriction:

Lemma 5.1. *If \mathbf{EL}^\dagger satisfies properties 1–7, A and B do not contain $!$, and $\mathbf{EL}^\dagger \vdash !q, A \rightarrow B$, then $\mathbf{EL}^\dagger \vdash !q \rightarrow A \setminus B$ and $\mathbf{EL}^\dagger \vdash !q \rightarrow B / A$.*

Proof.

$$\begin{array}{c}
\frac{q \rightarrow q \quad \frac{!q, A \rightarrow B}{A, !q \rightarrow B} \text{ (perm)}}{A / q, q, !q \rightarrow B} \text{ (/ } \rightarrow \text{)} \\
\frac{q, !q \rightarrow (A / q) \setminus B}{!q, !q \rightarrow (A / !q) \setminus B} \text{ (subst)} \quad \frac{A \rightarrow A}{A, !q \rightarrow A} \text{ (weak)} \\
\frac{!q, !q \rightarrow (A / !q) \setminus B}{!q \rightarrow (A / !q) \setminus B} \text{ (contr)} \quad \frac{A \rightarrow A / !q \quad B \rightarrow B}{(A / !q) \setminus B \rightarrow A \setminus B} \text{ (mon)} \\
\hline
!q \rightarrow A \setminus B \text{ (cut)}
\end{array}$$

The $/$ case is symmetric. \square

Theorem 3. *If $\Pi \rightarrow B$ is derivable in \mathbf{L}^* , and \mathbf{EL}^\dagger satisfies properties 1–7, then $!q, \Pi \rightarrow B$ is derivable in \mathbf{EL}^\dagger .*

Proof. Induction on derivation length. For the axiom case we use the weakening rule to add $!q$. Applications of Lambek rules are translated straightforwardly; the only non-trivial case is $(\rightarrow /)$ and $(\rightarrow \setminus)$ with an empty Π , where we use Lemma 5.1. \square

One could think that this effect is due to the weakening rule (this rule allows forcing the antecedent to be non-empty). However, in the fragment with only one variable a result like Theorem 3 can be achieved without weakening. Note that, in the view of [10], [11], [7, Chapter 3], [16], and [12], the one-variable fragment of the Lambek calculus is as powerful as the full calculus with a countable set of variables.

Theorem 4. *Let \mathbf{EL}^\dagger satisfy properties 1–4, 6, and 7. Let also contraction and permutation (but not weakening) rules be admissible in \mathbf{EL}^\dagger , and, in addition, let \mathbf{EL}^\dagger include the $(! \rightarrow)$ rule of the form*

$$\frac{\Delta_1, A, \Delta_2 \rightarrow C}{\Delta_1, !A, \Delta_2 \rightarrow C} (! \rightarrow),$$

if Δ_1 or Δ_2 contains a formula without occurrences of $!$ (this is the strongest version of Lambek’s restriction that could be imposed on this rule). In this case, if Π and B contain only one variable p and do not contain occurrences of $!$, and $\Pi \rightarrow B$ is derivable in \mathbf{L}^ , then $\Pi, !(p \setminus p) \rightarrow B$ is derivable in \mathbf{EL}^\dagger .*

Proof. First we state an easy technical lemma:

Lemma 5.2. *If A contains only one variable p and no occurrences of $!$, then $\mathbf{EL}^\dagger \vdash A, !(p \setminus p) \rightarrow A$.*

Proof. Induction on the complexity of A . If $A = p$, then $p, !(p \setminus p) \rightarrow p$ is derived as follows:

$$\frac{\frac{p \rightarrow p \quad p \rightarrow p}{p, p \setminus p \rightarrow p} (\setminus \rightarrow)}{p, !(p \setminus p) \rightarrow p} (! \rightarrow)$$

For $A = A_2 / A_1$ and $A = A_1 \setminus A_2$ we use the following derivations:

$$\frac{\frac{A_1 \rightarrow A_1 \quad A_2, !(p \setminus p) \rightarrow A_2}{A_2 / A_1, A_1, !(p \setminus p) \rightarrow A_2} (/ \rightarrow)}{\frac{A_2 / A_1, !(p \setminus p), A_1 \rightarrow A_2}{A_2 / A_1, !(p \setminus p) \rightarrow A_2 / A_1} (\rightarrow /)} \quad \frac{\frac{A_1 \rightarrow A_1 \quad A_2, !(p \setminus p) \rightarrow A_2}{A_1, A_1 \setminus A_2, !(p \setminus p) \rightarrow A_2} (\setminus \rightarrow)}{\frac{A_1 \setminus A_2, !(p \setminus p) \rightarrow A_1 \setminus A_2}{A_1 \setminus A_2, !(p \setminus p) \rightarrow A_1 \setminus A_2} (\rightarrow \setminus)}$$

$A_1 \rightarrow A_1$ is an axiom of \mathbf{L} (and, by property 1, is derivable in \mathbf{EL}^\dagger). $A_2, !(p \setminus p) \rightarrow A_2$ is derivable by induction hypothesis. \square

Then we proceed by induction on derivation. The axiom $A \rightarrow A$ becomes a derivable sequent $A, !(p \setminus p) \rightarrow A$ (Lemma 5.2). Now the only non-trivial case is to simulate $(\rightarrow /)$ and $(\rightarrow \setminus)$ with an empty Π :

$$\frac{\frac{\frac{p \rightarrow p \quad A, !(p \setminus p) \rightarrow B}{A / p, p, !(p \setminus p) \rightarrow B} (/ \rightarrow)}{p, !(p \setminus p) \rightarrow (A / p) \setminus B} (\rightarrow \setminus)}{\frac{!(p \setminus p), !(p \setminus p) \rightarrow (A / !(p \setminus p)) \setminus B}{!(p \setminus p) \rightarrow (A / !(p \setminus p)) \setminus B} (\text{subst})} \quad \frac{\frac{\text{by Lemma 5.2}}{A, !(p \setminus p) \rightarrow A} (\rightarrow /)}{\frac{A \rightarrow A / !(p \setminus p) \quad B \rightarrow B}{(A / !(p \setminus p)) \setminus B \rightarrow A \setminus B} (\text{mon})} \quad \frac{\frac{!(p \setminus p) \rightarrow (A / !(p \setminus p)) \setminus B}{!(p \setminus p) \rightarrow A \setminus B} (\text{contr})}{!(p \setminus p) \rightarrow A \setminus B} (\text{cut})$$

The $(\rightarrow /)$ case is handled symmetrically. \square

Theorems 3 and 4 show that there is no way to add the exponential modality to the Lambek calculus preserving Lambek's restriction, admissibility of (cut), and the substitution property at the same time.

In the next section we describe another yet extension of \mathbf{L} with the exponential modality. This extension features a version Lambek's restriction, admits substitution of formulae with $!$, but, on the other hand, only a limited version of the cut rule.

6 The 3rd Approach: \mathbf{EL}^{mk}

In order to restore type substitution as much as possible we consider the third approach to imposing Lambek's restriction on \mathbf{EL}^* . The trade-off here is that the cut rule is going to be admissible only in a limited form.

We present such a system in the form of *marked sequent calculus*. A marked sequent is an expression of the form $\Pi \rightarrow A$, where A is a type and Π is a sequence of pairs of the form $\langle B, m \rangle$, written as $B_{(m)}$, where B is a type and $m \in \{0, 1\}$ is the *marking bit*. A pair $B_{(0)}$ is called an *unmarked type*, and $B_{(1)}$ is called a *marked type*. The marking bits are utilized inside the derivation, and in the end they are forgotten, yielding a sequent in the original sense. If $\Gamma = B_{1(m_1)}, \dots, B_{k(m_k)}$, then by $!\Gamma$ we denote the sequence $(!B_1)_{(m_1)}, \dots, (!B_k)_{(m_k)}$.

Lambek's restriction is now formulated as follows: *every sequent should contain an unmarked type in the antecedent*.

The calculus \mathbf{EL}^{mk} is defined in the following way:

$$p_{(0)} \rightarrow p$$

$$\frac{\Pi, A_{(m)} \rightarrow B}{\Pi \rightarrow B / A} (\rightarrow /), \text{ where } \Pi \text{ contains an unmarked type}$$

$$\frac{A_{(m)}, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} (\rightarrow \setminus), \text{ where } \Pi \text{ contains an unmarked type}$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B_{(m)}, \Delta_2 \rightarrow C}{\Delta_1, (B / A)_{(m)}, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow) \quad \frac{\Pi \rightarrow A \quad \Delta_1, B_{(m)}, \Delta_2 \rightarrow C}{\Delta_1, \Pi, (A \setminus B)_{(m)}, \Delta_2 \rightarrow C} (\setminus \rightarrow)$$

$$\frac{\Delta_1, A_{(m)}, \Delta_2 \rightarrow B}{\Delta_1, (!A)_{(1)}, \Delta_2 \rightarrow B} (! \rightarrow), \text{ where } \Delta_1, \Delta_2 \text{ contains an unmarked type}$$

$$\frac{!\Gamma, \Delta \rightarrow A}{!\Gamma, !\Delta \rightarrow !A} (\rightarrow !) \quad \frac{\Delta_1, \Delta_2 \rightarrow A}{\Delta_1, (!A)_{(1)}, \Delta_2 \rightarrow A} (\text{weak})$$

$$\frac{(!A)_{(m_1)}, (!A)_{(m_2)}, \Delta \rightarrow B}{(!A)_{(\min\{m_1, m_2\})}, \Delta \rightarrow B} (\text{contr})$$

$$\frac{\Delta_1, B_{(m_2)}, (!A)_{(m_1)}, \Delta_2 \rightarrow C}{\Delta_1, (!A)_{(m_1)}, B_{(m_2)}, \Delta_2 \rightarrow C} (\text{perm}_1) \quad \frac{\Delta_1, (!B)_{(m_2)}, A_{(m_1)}, \Delta_2 \rightarrow C}{\Delta_1, A_{(m_1)}, (!B)_{(m_2)}, \Delta_2 \rightarrow C} (\text{perm}_2)$$

Recall that all proofs are cut-free. Also note that in **EL** we use a stronger form of the $(\rightarrow !)$ rule. In **EL*** this new rule could be simulated by applying the $(! \rightarrow)$ rule for all formulae in Δ and then using the original $(\rightarrow !)$ rule, but here the $(! \rightarrow)$ rule will fail to satisfy the restriction.

The substitution property is now formulated as follows:

Theorem 5. *Let $A[q := Q]$ (resp., $\Pi[q := Q]$) be the result of substituting Q for q in type A (resp., marked sequence Π). Then $\mathbf{EL}^{\text{mk}} \vdash \Pi \rightarrow A$ implies $\mathbf{EL}^{\text{mk}} \vdash \Pi[q := Q] \rightarrow A[q := Q]$.*

Proof. By structural induction on Q we prove that $\mathbf{EL}^{\text{mk}} \vdash Q_{(0)} \rightarrow Q$ for every type Q . Then we just replace q with Q everywhere in the proof. \square

The cut rule in \mathbf{EL}^{mk} is generally not admissible: the sequents $(!q)_{(0)} \rightarrow (p / !q) \setminus p$ and $((p / !q) \setminus p)_{(0)} \rightarrow p \setminus p$ are derivable in \mathbf{EL}^{mk} , but $(!q)_{(0)} \rightarrow p \setminus p$ is not. This counterexample is actually taken from the proof of Theorem 3.

The cut rule is admissible only in the following limited version:

Theorem 6. *If $\mathbf{EL}^{\text{mk}} \vdash \Pi \rightarrow A$, $\mathbf{EL}^{\text{mk}} \vdash \Delta_1, A_{(0)}, \Delta_2 \rightarrow C$, and A doesn't contain $!$, then $\mathbf{EL}^{\text{mk}} \vdash \Delta_1, \Pi, \Delta_2 \rightarrow C$.*

This theorem is proved using the standard argument, just as for **L**.

Compare \mathbf{EL}^{mk} with \mathbf{EL}^- . These two systems are not connected with any strong form of conservativity or equivalence: on one hand, the sequent $!r, r \setminus !p, !(p \setminus q) \rightarrow q$ is derivable in \mathbf{EL}^- , but not in \mathbf{EL}^{mk} ; on the other hand, for $!p, !(p \setminus q) \rightarrow q$ the situation is opposite. Fortunately, the following holds:

Lemma 6.1. *If Γ , Π , and A do not contain $!$, then*

$$\mathbf{EL}^{\text{mk}} \vdash !\Gamma, \Pi \rightarrow A \iff \mathbf{EL}^- \vdash !\Gamma, \Pi \rightarrow A.$$

Proof. Since for a sequent of the form $!\Gamma, \Pi \rightarrow A$ the rule $(\rightarrow !)$ can never appear in the proof, marked types in the antecedent are exactly the types starting with $!$, and the two versions of Lambek's restriction coincide. \square

7 Conservativity over **L**

The three calculi defined above are conservative over **L**:

Proposition 7. *If Π and A do not contain $!$, then*

$$\mathbf{L} \vdash \Pi \rightarrow A \iff \mathbf{EL}^{\text{wk}} \vdash \Pi \rightarrow A \iff \mathbf{EL}^- \vdash \Pi \rightarrow A \iff \mathbf{EL}^{\text{mk}} \vdash \Pi \rightarrow A$$

(for \mathbf{EL}^{mk} , all types in Π get the 0 marking bit).

Note that Π is necessarily non-empty.

Therefore, we guarantee that in all approaches the innovation affects only the new exponential connective, and keeps the original Lambek system intact. For \mathbf{EL}^- and \mathbf{EL}^{mk} adding fresh exponentials to the antecedent also doesn't affect Lambek's restriction:

Proposition 8. *If Π and A do not contain $!$, and p is a variable not occurring in Π and A , then*

$$\mathbf{EL}^- \vdash !p, \Pi \rightarrow A \iff \mathbf{EL}^{\text{mk}} \vdash !p, \Pi \rightarrow A \iff \mathbf{L} \vdash \Pi \rightarrow A$$

(for the \mathbf{EL}^{mk} case, $!p$ gets marking bit 1 and types from Π get 0).

For \mathbf{EL}^{wk} , due to Lemma 2.1, the situation is different: if Π and A don't contain $!$, and p is a fresh variable, then

$$\mathbf{EL}^{\text{wk}} \vdash !p, \Pi \rightarrow A \iff \mathbf{EL}^* \vdash \Pi \rightarrow A \iff \mathbf{L}^* \vdash \Pi \rightarrow A.$$

Recall that, for example, $(q \setminus q) \setminus p \rightarrow p$ is derivable in \mathbf{L}^* , but not in \mathbf{L} .

8 Generative Grammars and the Lambek Calculus with Non-Logical Axioms

In this subsection we introduce *axiomatic extensions* of the Lambek calculus \mathbf{L} , following [3]. These extensions are going to be useful for proving undecidability results à la Theorem 1.

Let \mathcal{A} be a set of sequents. Then by $\mathbf{L} + \mathcal{A}$ we denote \mathbf{L} augmented with sequents from \mathcal{A} as new axioms and also the cut rule (which is no longer eliminable). Elements of \mathcal{A} are called *non-logical axioms*.

Further we consider non-logical axioms of a special form: either $p, q \rightarrow r$, or $p / q \rightarrow r$, where p, q, r are variables. Buszkowski calls them *special* non-logical axioms. In this case, $\mathbf{L} + \mathcal{A}$ can be formulated in a cut-free way [3]: instead of non-logical axioms of the form $p, q \rightarrow r$ or $p / q \rightarrow r$ we use rules

$$\frac{\Pi_1 \rightarrow p \quad \Pi_2 \rightarrow q}{\Pi_1, \Pi_2 \rightarrow r} (\text{red}_1) \quad \text{and} \quad \frac{\Pi, q \rightarrow p}{\Pi \rightarrow r} (\text{red}_2), \text{ where } \Pi \neq \Lambda$$

respectively. This calculus admits the cut rule [3]. Further we'll mean it when talking about $\mathbf{L} + \mathcal{A}$. We'll use the term *Buszkowski's rules* for (red_i) .

Now we define two notions of *formal grammar*. The first one is the widely known formalism of *generative grammars* introduced by Chomsky. If

Σ is an alphabet (i.e. a finite non-empty set), then by Σ^* we denote the set of all words over Σ (including the empty word). A generative grammar is a quadruple $G = \langle N, \Sigma, s, P \rangle$, where N and Σ are two disjoint alphabets, $s \in N$, and P is a set or *rules*. Here we consider only rules of two forms: $x \rightarrow y_1 y_2$ or $x_1 x_2 \rightarrow y$, where $x, y, x_i, y_i \in N \cup \Sigma$. If $v = u_1 \alpha u_2$, $w = u_1 \beta u_2$, and $(\alpha \rightarrow \beta) \in P$, then this rule can be *applied* to v yielding w : $v \Rightarrow w$. By \Rightarrow^* we denote the reflexive and transitive closure of \Rightarrow . Finally, the *language generated by G* is the set of all words $w \in \Sigma^*$, such that $s \Rightarrow^* w$. Note that the empty word can't be produced by a generative grammar as defined above.

It is well known that the class of languages generated by generative grammars coincides with the class of all recursively enumerable (r.e.) languages without the empty word.

The second family of formal grammar we are going to consider is the class of *Lambek categorial grammars with non-logical axioms*. A Lambek grammar is a tuple $\mathcal{G} = \langle \Sigma, \mathcal{A}, H, \triangleright \rangle$, where Σ is an alphabet, \mathcal{A} is a set of non-logical axioms, H is a type, and $\triangleright \subseteq \text{Tp} \times \Sigma$ is a finite binary correspondence between types and letter, called *type assignment*. A word $w = a_1 \dots a_n$ belongs to the language generated by \mathcal{G} iff there exist such types A_1, \dots, A_n that $A_i \triangleright a_i$ ($i = 1, \dots, n$) and $\mathbf{L} + \mathcal{A} \vdash A_1, \dots, A_n \rightarrow H$.

If we use $\mathbf{L}_/$ instead of \mathbf{L} , we get the notion of $\mathbf{L}_/$ -grammar with non-logical axioms. It's easy to see that all languages generated by Lambek grammars are r.e., therefore, they can be generated by generative grammars. Buszkowski [3] proves the converse:

Theorem 9. *Every language generated by a generative grammar can be generated by an $\mathbf{L}_/$ -grammar with special non-logical axioms.*

In comparison, for $\mathcal{A} = \emptyset$ Pentus' theorem [22] states that all languages generated are context-free. Thus, even simple (special) non-logical axioms dramatically increase the power (and complexity) of Lambek grammars.

Since there exist undecidable r.e. languages, Buszkowski obtains the following [3]:

Theorem 10. *There exists such \mathcal{A} that the derivability problem for $\mathbf{L}_/ + \mathcal{A}$ is undecidable.*

9 Undecidability of \mathbf{EL}^- and \mathbf{EL}^{mk}

Recall that \mathbf{EL}^- , defined in Section 3, involves two division operations, and no product. The calculus $\mathbf{EL}_/^-$ is the fragment of \mathbf{EL}^- , where we confine

ourselves only to the right division.

Theorem 11. *The derivability problem for \mathbf{EL}^- and even for $\mathbf{EL}_/^-$ is undecidable.*

We take a set \mathcal{A} of non-logical axioms of the forms $p, q \rightarrow r$ or $p / q \rightarrow r$ and encode them in \mathbf{EL}^- using the exponential. Let $\mathcal{G}_{\mathcal{A}} = \{(r / q) / p \mid (p, q \rightarrow r) \in \mathcal{A}\} \cup \{r / (p / q) \mid (p / q \rightarrow r) \in \mathcal{A}\}$ and let $\Gamma_{\mathcal{A}}$ be a sequence of all types from $\mathcal{G}_{\mathcal{A}}$ in any order. Then the following holds:

Lemma 9.1. $\mathbf{L}_/ + \mathcal{A} \vdash \Pi \rightarrow A \iff \mathbf{EL}_/^- \vdash !\Gamma_{\mathcal{A}}, \Pi \rightarrow A$.

Proof. $\boxed{\Rightarrow}$ Proceed by induction on the derivation of $\Pi \rightarrow A$ in $\mathbf{L}_/ + \mathcal{A}$. If $\Pi \rightarrow A$ is an axiom of the form $A \rightarrow A$, then we get $\mathbf{EL}_/^- \vdash !\Gamma_{\mathcal{A}}, A \rightarrow A$ by application of the (weak) rule.

If $A = B / C$, and $\Pi \rightarrow A$ is obtained using the $(\rightarrow /)$ rule, then $!\Gamma_{\mathcal{A}}, \Pi \rightarrow A$ is derived using the same rule:

$$\frac{!\Gamma_{\mathcal{A}}, \Pi, C \rightarrow B}{!\Gamma_{\mathcal{A}}, \Pi \rightarrow B / C}$$

Here Π is not empty, and consists of non-bang-formulae, therefore the application of this rule is eligible in $\mathbf{EL}_/^-$; $\mathbf{EL}_/^- \vdash !\Gamma_{\mathcal{A}}, \Pi, C \rightarrow B$ by induction hypothesis.

If $\Pi = \Phi_1, B / C, \Psi, \Phi_2$, and $\Pi \rightarrow A$ is obtained by $(/ \rightarrow)$ from $\Psi \rightarrow C$ and $\Phi_1, B, \Phi_2 \rightarrow A$, then for $!\Gamma_{\mathcal{A}}, \Pi \rightarrow A$ we have the following derivation in $\mathbf{EL}_/^-$, where $*$ means several applications of the rules in any order.

$$\frac{\frac{\frac{!\Gamma_{\mathcal{A}}, \Psi \rightarrow C \quad !\Gamma_{\mathcal{A}}, \Phi_1, B, \Phi_2 \rightarrow A}{!\Gamma_{\mathcal{A}}, \Phi_1, B / C, !\Gamma_{\mathcal{A}}, \Psi, \Phi_2 \rightarrow A} (/ \rightarrow)}{!\Gamma_{\mathcal{A}}, !\Gamma_{\mathcal{A}}, \Phi_1, B / C, \Psi, \Phi_2 \rightarrow A} (\text{perm}_1)^*}{!\Gamma_{\mathcal{A}}, \Phi_1, B / C, \Psi, \Phi_2 \rightarrow A} (\text{contr}, \text{perm}_1)^*$$

Finally, $\Pi \rightarrow A$ can be obtained by application of Buszkowski's rules (red_1) or (red_2). In the first case, $A = r$, $\Pi = \Pi_1, \Pi_2$; $\mathbf{L} + \mathcal{A} \vdash \Pi_1 \rightarrow p$, and $\mathbf{L} + \mathcal{A} \vdash \Pi_2 \rightarrow q$. Furthermore, $\mathcal{G}_{\mathcal{A}} \ni (r / q) / p$, thus we get the following derivation in $\mathbf{EL}_/^-$:

$$\frac{\frac{\frac{!\Gamma_{\mathcal{A}}, \Pi_2 \rightarrow q \quad r \rightarrow r}{r / q, !\Gamma_{\mathcal{A}}, \Pi_2 \rightarrow r} (/ \rightarrow)}{!(r / q) / p, !\Gamma_{\mathcal{A}}, \Pi_1, !\Gamma_{\mathcal{A}}, \Pi_2 \rightarrow r} (/ \rightarrow)}{!(r / q) / p, !\Gamma_{\mathcal{A}}, \Pi_1, !\Gamma_{\mathcal{A}}, \Pi_2 \rightarrow r} (! \rightarrow)}{!\Gamma_{\mathcal{A}}, !((r / q) / p), \Pi_1, \Pi_2 \rightarrow r} (\text{contr}, \text{perm}_1)^*}{!\Gamma_{\mathcal{A}}, !((r / q) / p), \Pi_1, \Pi_2 \rightarrow r} (\text{contr}, \text{perm}_1)^*$$

The application of $(! \rightarrow)$ here is legal, since Π_1 and Π_2 are non-empty and consist of non-bang-formulae.

In the (red_2) case, $A = r$, and we have $!\Gamma_{\mathcal{A}}, \Pi, q \rightarrow p$ in the induction hypothesis. Again, $\mathcal{G}_{\mathcal{A}} \ni r / (p / q)$, and we proceed like this:

$$\frac{\frac{\frac{!\Gamma_{\mathcal{A}}, \Pi, q \rightarrow p}{!\Gamma_{\mathcal{A}}, \Pi \rightarrow p / q} (\rightarrow /)}{r / (p / q), !\Gamma_{\mathcal{A}}, \Pi \rightarrow r} (\rightarrow /)}{\frac{!(r / (p / q)), !\Gamma_{\mathcal{A}}, \Pi \rightarrow r}{!\Gamma_{\mathcal{A}}, \Pi \rightarrow r} (! \rightarrow)} (\text{contr, perm}_1)^*$$

Here, again, Π is not empty and consists of non-bang-formulae, therefore we can legally apply $(! \rightarrow)$ and $(\rightarrow /)$.

$\boxed{\Leftarrow}$ For deriving sequents of the form $!\Gamma, \Pi \rightarrow A$, where Γ , Π , and A do not contain the exponential, one can use a simpler calculus than $\mathbf{EL}^-_{/}$:

$$\begin{array}{c} !\Gamma, p \rightarrow p \\ \\ \frac{!\Gamma, \Pi, B \rightarrow A}{!\Gamma, \Pi \rightarrow A / B} (\rightarrow /), \text{ where } \Pi \neq \Lambda \quad \frac{!\Gamma, \Pi \rightarrow B \quad !\Gamma, \Delta_1, A, \Delta_2 \rightarrow C}{!\Gamma, \Delta_1, A / B, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow) \\ \\ \frac{!\Gamma, \Delta_1, B, \Delta_2 \rightarrow A}{!\Gamma, \Delta_1, \Delta_2 \rightarrow A} (! \rightarrow), \text{ where } B \text{ is a type from } \Gamma \text{ and } \Delta_1, \Delta_2 \neq \Lambda \end{array}$$

Here (weak) is hidden into the axiom, (contr) comes within $(\rightarrow /)$, and $(! \rightarrow)$ includes both (perm_i) and (contr) in the needed form. One can easily see that if $\mathbf{EL}^-_{/} \vdash !\Gamma, \Pi \rightarrow A$, where Γ , Π , and A do not contain $!$, then this sequent is derivable in the simplified calculus. Moreover, the $(! \rightarrow)$ rule is interchangeable with the others in the following ways:

$$\frac{\frac{!\Gamma, \Delta_1, C, \Delta_2, B \rightarrow A}{!\Gamma, \Delta_1, C, \Delta_2 \rightarrow A / B} (\rightarrow /)}{!\Gamma, \Delta_1, \Delta_2 \rightarrow A / B} (! \rightarrow) \quad \rightsquigarrow \quad \frac{!\Gamma, \Delta_1, C, \Delta_2, B \rightarrow A}{!\Gamma, \Delta_1, \Delta_2, B \rightarrow A} (! \rightarrow)}{!\Gamma, \Delta_1, \Delta_2 \rightarrow A / B} (\rightarrow /)$$

$$\frac{\frac{!\Gamma, \Pi \rightarrow B \quad !\Gamma, \Delta_1, A, \Delta'_2, D, \Delta''_2 \rightarrow C}{!\Gamma, \Delta_1, A / B, \Pi, \Delta'_2, D, \Delta''_2 \rightarrow C} (/ \rightarrow)}{!\Gamma, \Delta_1, A / B, \Pi, \Delta'_2, \Delta''_2 \rightarrow C} (! \rightarrow)$$

\Downarrow

$$\frac{! \Gamma, \Pi \rightarrow B \quad \frac{! \Gamma, \Delta_1, A, \Delta'_2, D, \Delta''_2 \rightarrow C}{! \Gamma, \Delta_1, A, \Delta'_2, \Delta''_2 \rightarrow C} (! \rightarrow)}{! \Gamma, \Delta_1, A / B, \Pi, \Delta'_2, \Delta''_2 \rightarrow C} (/ \rightarrow)$$

And the same, if D appears inside Δ_1 or Π . Finally, consecutive applications of $(! \rightarrow)$ are always interchangeable.

After applying these transformations, we achieve a derivation where $(! \rightarrow)$ is applied immediately after applying $(/ \rightarrow)$ with the same active type (the other case, when it is applied after the axiom to p , is impossible, since then it violates the non-emptiness condition). In other words, applications of $(! \rightarrow)$ appear only in the following two situations:

$$\frac{\frac{! \Gamma, \Pi \rightarrow p \quad ! \Gamma, \Delta_1, r / q, \Delta_2 \rightarrow A}{! \Gamma, \Delta_1, (r / q) / p, \Pi, \Delta_2 \rightarrow A} (/ \rightarrow)}{! \Gamma, \Delta_1, \Pi, \Delta_2 \rightarrow A} (! \rightarrow)$$

and

$$\frac{\frac{! \Gamma, \Pi \rightarrow p / q \quad ! \Gamma, \Delta_1, r, \Delta_2 \rightarrow A}{! \Gamma, \Delta_1, r / (p / q), \Pi, \Delta_2 \rightarrow A} (/ \rightarrow)}{! \Gamma, \Delta_1, \Pi, \Delta_2 \rightarrow A} (! \rightarrow)$$

Now we prove the statement $\mathbf{EL}^- / \vdash ! \Gamma_{\mathcal{A}}, \Pi \rightarrow A \Rightarrow \mathbf{L} + \mathcal{A} \vdash \Pi \rightarrow A$ by induction on the above canonical derivation. If $! \Gamma_{\mathcal{A}}, \Pi \rightarrow A$ is an axiom or is obtained by an application of $(/ \rightarrow)$ or $(\rightarrow /)$, we apply the corresponding rules in $\mathbf{L} + \mathcal{A}$, so the only interesting case is $(! \rightarrow)$. Consider the two possible situations.

In the $(r / q) / p$ case, by induction hypothesis we get $\mathbf{L} + \mathcal{A} \vdash \Pi \rightarrow p$ and $\mathbf{L} + \mathcal{A} \vdash \Delta_1, r / q, \Delta_2 \rightarrow A$, and then we develop the following derivation in $\mathbf{L} + \mathcal{A}$ (recall that (cut) is admissible there):

$$\frac{\Pi \rightarrow p \quad \frac{\frac{p, q \rightarrow r}{p \rightarrow r / q} (\rightarrow /) \quad \Delta_1, r / q, \Delta_2 \rightarrow A}{\Delta_1, p, \Delta_2 \rightarrow A} (\text{cut})}{\Delta_1, \Pi, \Delta_2 \rightarrow A} (\text{cut})$$

In the case of $r / (p / q)$, the derivation looks like this:

$$\frac{\Pi \rightarrow p / q \quad \frac{p / q \rightarrow r \quad \Delta_1, r, \Delta_2 \rightarrow A}{\Delta_1, p / q, \Delta_2 \rightarrow A} (\text{cut})}{\Delta_1, \Pi, \Delta_2 \rightarrow A} (\text{cut})$$

□

Note that in this proof we don't need any form of the cut rule for \mathbf{EL}^- .

Theorem 12. *The derivability problem for $\mathbf{EL}_/^{\text{mk}}$ (and, thus, for \mathbf{EL}^{mk}) is undecidable.*

Proof. By Lemma 6.1 we have that if Γ , Π , and A do not contain $!$, then $\mathbf{EL}_/^{\text{mk}} \vdash !\Gamma, \Pi \rightarrow A \iff \mathbf{EL}_/^- \vdash !\Gamma, \Pi \rightarrow A$. \square

Of course, everything discussed above can be dually performed for \backslash instead of $/$, yielding undecidability for $\mathbf{EL}_{\backslash}^{\text{mk}}$ and $\mathbf{EL}_{\backslash}^-$.

10 Conclusion

The derivability problem for the original Lambek calculus, without exponential modalities, is decidable and belongs to the NP class. This happens because the cut-free proof of a sequent has linear size with respect to the sequent's length. For the full Lambek calculus [23] and for its fragments with any two of three connectives (two divisions [25] or one division and the product [26]) the derivability problem is NP-complete.

On the other hand, for derivability problem in $\mathbf{L}_/$ there exists a polynomial time algorithm [24]. Thus the one-division fragment of the Lambek calculus appears to be significantly simpler. Despite this, in our undecidability results for \mathbf{EL}^- and \mathbf{EL}^{mk} we use only one of the two divisions.

Future Work

It appears that the technique used in the $\boxed{\Leftarrow}$ part of the proof of Lemma 9.1 is an instance of *focusing* [2][20] in the non-commutative situation, which should be investigated systematically and in detail.

We also plan to investigate calculi with modalities where not all of the structural rules ((weak), (perm_i), and (contr)) are kept. Once we remove (contr), the derivability problem becomes decidable and falls into the NP class. The interesting question is to determine precise complexity bounds (P or NP-hard) for the fragments with only one division and bang (if we have at least two of the three Lambek connectives, even the calculus without bang is NP-complete). The variants where we have only permutational rules (both (perm_1) and (perm_2), or only one of them) are particularly interesting for linguistic applications (see, for example, [17][18]). The other question here is (un)decidability of the variant without (perm_i) (only (weak) and (contr)).

In the case of commutative linear logic Nigam and Miller [21] consider calculi that have several modalities interacting with each other, and different

modalities are controlled by different sets of structural rules. These modalities are called *subexponentials*. We plan to study subexponentials in the non-commutative case under the umbrella of the Lambek calculus.

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